Applications of Lobachevsky Geometry to the Relativistic Two-Body Problem[1]

Plamen P. Fiziev [2]

Department of Theoretical Physics, Sofia University,

5 James Bourchier Boulevard, BG-1164, Sofia, Bulgaria

Abstract

In this talk we consider the geometrical basis for the reduction of the relativistic 2-body problem, much like the non-relativistic one, to describing the motion of an effective particle in an external field. It is shown that this possibility is deeply related with the Lobachevsky geometry. The concept of relativistic reduced mass and effective relativistic particle is discussed using this geometry. Different recent examples for application of relativistic effective particle are described in short.

1 Introduction

The classical (non-quantum) relativistic two-body problem has a long history. Although at present many different approaches to this problem exist, we still do not have a satisfactory solution in many cases of different interactions between the two massive point particles. Much has been done in the case of relativistic interaction of two electrically charged particles [1, 2, 3, 4]. An essential progress is reached in the case of two-particle gravitational interaction [1, 2, 4], in the case of two massive

[1]Talk given at the International seminar: *Application and Development of Lobachevsky Ideas in Modern Physics*, dedicated to 75-th birthday of Professor N. A. Chernikov, Dubna, 25-27 February 2004

[2]E-mail: fiziev@phys.uni-sofia.bg

relativistic particles with string interaction between them [5] and in the case of areal interaction [6].

There exist several basic obstacles to general solution of the problem: the existence of many different possible choices of the time variable, the absence of a unique choice of the center-of-mass position variables, the complications, which appear in the attempts to introduce a proper phase-space of relativistic two-particles and during the reduction of their degrees of freedom, the correct inclusion of relativistic retardation and corresponding non-locality of the interactions, the nonlinear character of typical relativistic interactions between particles, like gravitational interaction and string one, the problem how to take account of the internal spin both of the interacting particles and of the carrier of the interaction, etc.

In the present talk we will concentrate on one of the possible approaches to the relativistic two-particle problem, which is based on the introduction of an effective particle – *the relativistic reduced mass*. This approach is analogous to the non-relativistic one and was introduced in a heuristic way at first in the quantum relativistic problems by I. Todorov [7].

A new derivation of the basic notions of the relativistic effective one-particle approach to the relativistic two-particle problem and some applications were given in [8]. It turns out that this derivation lies essentially on the Lobachevsky geometry in the velocity space of the relativistic particles. In the present talk we will give a more detailed derivation of the notion of relativistic effective particle, based on this geometry. We also review in short some of the recent developments and applications of this idea [9].

2 The Kinematics of One Relativistic Massive Particle and Lobachevsky Geometry

In this section we remind the reader the basic notions of the simple kinematics of one *free* relativistic particle with mass $m > 0$ and its relation with Lobachevsky geometry in the velocity space .

In this section we choose the laboratory frame system K and and laboratory time t. Further on we use the units in which $c = 1$.

2.1 The Kinematics of One Free Relativistic Massive Particle

The 3D position of a point particle is described by its radius-vector $\vec{x}$ and 3D-velocity $\vec{v}$. We use a standard notations $\{x^0, \vec{x}\}$ ($x^0 = t$) for the Cartesian coordinates in the 4D pseudo-Euclidean space-time $\mathbb{E}^{(1,3)}$ and a standard action $\mathcal{A}_m = -m \int ds = -m \int \sqrt{1 - \vec{v}^{\,2}} dt$. The 4D canonical momentum p has components $\{p_0, \vec{p}\}$:

$$p_0 = \frac{m}{\sqrt{1 - \vec{v}^{\,2}}}, \ \ \vec{p} = \frac{m\vec{v}}{\sqrt{1 - \vec{v}^{\,2}}}. \tag{1}$$

According to our convention the 4D velocity is $u = p/\sqrt{-p^2}$. As a consequences one obtains

$$p = mu, \quad p^2 = \vec{p}^{\,2} - p_0^2 \stackrel{w}{=} -m^2, \quad u^2 = \vec{u}^{\,2} - u_0^2 \stackrel{w}{=} -1. \tag{2}$$

Here the simbol "$\stackrel{w}{=}$" denotes a "weak" equality, i.e. a constraint in Dirac's sense [10]. These relations define the relativistic 4D "hyperboloid of velocities". In addition we have for the 3D velocities the useful relations

$$\vec{u} = \frac{\vec{v}}{\sqrt{1 - \vec{v}^{\,2}}}, \quad \vec{v} = \frac{\vec{u}}{\sqrt{1 + \vec{u}^{\,2}}}, \quad \left(1 - \vec{v}^{\,2}\right)\left(1 + \vec{u}^{\,2}\right) = 1. \tag{3}$$

Let K^U is an inertial frame which moves with respect to the laboratory frame K with a 4D velocity U. Then the velocity of the mass m with respect the system

K^U is

$$u^U = \Lambda_U^{-1} u, \tag{4}$$

where $\Lambda_U \in SO(1,3)$ is a pure Lorentz transformation that carries the quantities from frame K into the frame K^U. The conditions $\Lambda^\mu{}_\nu U^\nu = \delta_0^\mu$ and positive definiteness determine the (symmetric) Lorentzian matrix Λ uniquely:

$$\Lambda = \begin{pmatrix} U^0 & -U_j \\ -U^i & \delta_j^i + \frac{U^i U_j}{1+U^0} \end{pmatrix}. \tag{5}$$

Here and further on the Latin indexes take values $i, j, ... = 1, 2, 3$; and the Greek ones – the values $\alpha, \beta, ... = 0, 1, 2, 3$; .

The formula (4) is analogous to the corresponding Galileo formula for superposition of velocities in non-relativistic mechanics:

$$\vec{v} \to \vec{v}^{\,V} = g_V^{-1}(\vec{v}) = \vec{v} + \vec{V}. \tag{6}$$

It describes a Galileo's translation in the velocity space $g_V^{-1} \in G_{10}$, i.e. the transition from laboratory frame K to a moving inertial frame K^V. Here G_{10} is the Galileo's group.

In a transparent form the relation (4) gives

$$\begin{aligned} u_0^U &= U_0 u_0 - \vec{U} \cdot \vec{u}, \\ \vec{u}^{\,U} &= U_0\, \vec{u} - u_0\, \vec{U} + \frac{\vec{U} \times \left(\vec{U} \times \vec{u}\right)}{1+U_0} = U_0\, \vec{u}_{||} - u_0\, \vec{U} + \vec{u}_\perp. \end{aligned} \tag{7}$$

Here

$$\vec{u}_{||} = \mathcal{P}_U \vec{u},$$

$$\vec{u}_\perp = \mathcal{P}_U^\perp \vec{u},$$

where

$$\mathcal{P}_U = \vec{U} \otimes \vec{U} / U^2$$

and

$$\mathcal{P}_U^\perp = Id_3 - \vec{U} \otimes \vec{U} / U^2$$

are the corresponding projector operators, and Id_3 is the 3D identity operator.

Using the 3D velocity $\vec{V}$ we obtain for the 4D velocity $U = \left\{\frac{1}{\sqrt{1-\vec{V}^{\,2}}}, \frac{\vec{V}}{\sqrt{1-\vec{V}^{\,2}}}\right\}$ and the standard form of Lorenz transformation:

$$u_0^U = \frac{u_0 - \vec{V}\cdot\vec{u}}{\sqrt{1-\vec{V}^{\,2}}}, \quad \vec{u}^{\,U} = \frac{\vec{u}_{||} - u_0\vec{V}}{\sqrt{1-\vec{V}^{\,2}}} + \vec{u}_{\perp}. \tag{8}$$

For the transformation of the 4D momentum one obtains an analogous relations:

$$p^U = \Lambda_U^{-1} p, \tag{9}$$

$$p_0^U = U_0\, p_0 - \vec{U}\cdot\vec{p}, \quad \vec{p}^{\,U} = U_0\,\vec{p} - p_0\,\vec{U} + \frac{\vec{U}\times\left(\vec{U}\times\vec{p}\right)}{1+U_0} = U_0\,\vec{p}_{||} - p_0\,\vec{U} + \vec{p}_{\perp}. \tag{10}$$

As we see, we have a point transformation in the momentum space, which can be easily extended to a canonical transformation on the whole phase space $\mathbb{M}^{(8)}_{p,x}$: $\{p, x\} \to \{p, x\}$ of the type $x\,dp + p^U dx^U = dF(p, x^U)$ with a generating function

$$F(p, x^U) = \vec{p}_{\perp}\cdot\vec{x}^{\,U} + U_0\,\vec{p}_{||}\cdot\vec{x}^{\,U} - p_0\,\vec{U}\cdot\vec{x}^{\,U} - \left(U_0\,p_0 - \vec{U}\cdot\vec{p}\right)(x^0)^U. \tag{11}$$

Using the formulas $x^0 = -\partial_{p_0} F$ and $\vec{x} = \partial_{\vec{p}} F$ one easily obtains the standard Lorenz transformations:

$$\begin{aligned}
x^0 = U_0(x^0)^U + \vec{U}\cdot\vec{x}^{\,U} &= \frac{(x^0)^U + \vec{V}\cdot\vec{x}^{\,U}}{\sqrt{1-\vec{V}^{\,2}}},\\
\vec{x} = (\vec{x}_{\perp})^U + U_0(\vec{x}_{||})^U + (x_0)^U\vec{U} = (\vec{x}_{\perp})^U &+ \frac{(\vec{x}_{||})^U + (x^0)^U\vec{V}}{\sqrt{1-\vec{V}^{\,2}}};\\
(x^0)^U &= \frac{(x^0) - \vec{V}\cdot\vec{x}}{\sqrt{1-\vec{V}^{\,2}}},\\
\vec{x} = \vec{x}_{\perp} &+ \frac{(\vec{x}_{||}) - x^0\vec{V}}{\sqrt{1-\vec{V}^{\,2}}}.
\end{aligned} \tag{12}$$

2.2 The Lobachevsky Geometry in the Velocity Space

Following Fock [1], let us consider an infinitesimal change of the velocity $\vec{U} = \vec{u} + d\vec{u}$. Then from the representations $\vec{u}^{\,U} = \vec{u} + d\vec{u}^{\,U}$, $d\vec{u} = d\vec{u}_{\perp} + d\vec{u}_{||}/u_0$, where $\vec{u}_{||} =$

$(\vec{u}\cdot d\vec{u}/\vec{u}^2)\vec{u}$, $d\vec{u}_\perp = d\vec{u} - d\vec{u}$ one easily obtains the metric in the velocity space in the form:

$$d\sigma^2 = (d\vec{u}^{\,U})^2 = g_{ij}(d\vec{u}^{\,U})^i(d\vec{u}^{\,U})^j, \tag{13}$$

where the metric tensor is given by the 3×3 matrix

$$g(\vec{u}) = Id_3 - \frac{\vec{u}\otimes\vec{u}}{1+\vec{u}^{\,2}}. \tag{14}$$

In spherical coordinates $\vec{u} = \{\rho_u\cos\theta\cos\varphi, \rho_u\cos\theta\sin\varphi, \rho_u\sin\theta\}$ the metric in the velocity space reads:

$$d\sigma^2 = \frac{(d\rho_u)^2}{1+\rho_u^2} + \rho_u^2\left((d\theta)^2 + \sin^2\theta(d\varphi)^2\right). \tag{15}$$

This formula shows explicitly that the $\vec{u}$-space is a Riemannian one with a constant negative curvature -1, i.e. a Lobachevsky space. It is just the hyperboloid $u^2 = -1$ in the 4D pseudo-Euclidean space-time $\mathbb{E}^{(1,3)}$ and its metric (13) is just the restriction:

$$d\sigma^2 = \Big((d\vec{u})^2 - (du_0)^2\Big)_{u_0=\pm\sqrt{1+\vec{u}^{\,2}}}.$$

The length $\sigma(u,U)$ of a geodesic line, which connects two points u and U on this hyperboloid, is defined by the formulas:

$$\begin{aligned}
\sinh(\sigma(u,U)) &= \sqrt{\left(U_0\,\vec{u} - u_0\,\vec{U}\right)^2 - \left(\vec{U}\times\vec{u}\right)^2}, \\
\cosh(\sigma(u,U)) &= U_0\,u_0 - \vec{U}\cdot\vec{u}\ .
\end{aligned} \tag{16}$$

3 Effective Particles in the Two Particle Problem

3.1 The Non-Relativistic Two Particle Problem

It is well known from the textbooks on classical mechanics, that one can reduce the two-particle problem to one-particle one, introducing two effective particles: the center of mass (CM) and the reduced mass m. In this section we give some

nonstandard way of introducing a reduced mass in the non-relativistic case. Our specific approach is based on the momentum space considerations and allows a proper relativistic generalization.

In the non-relativistic mechanics the CM is an effective particle with mass $M := m_1 + m_2$, where $m_{1,2}$ are the rest masses of the physical point particles at 3D positions with radius-vectors $\vec{r}_{1,2}$ and 3D velocities $\vec{v}_{1,2}$. It is supposed that the particles are moving in vacuum, i.e., without influence of any external forces, but they can interact between themselves. The position vector of the CM and its velocity in the laboratory frame K are

$$\vec{R} := \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2, \quad \vec{V} = \frac{m_1}{M}\vec{v}_1 + \frac{m_2}{M}\vec{v}_2. \tag{17}$$

By definition the CM-frame (CMF) is the inertial frame in which $R \equiv 0$, $V \equiv 0$. The velocities of the particles in CMF are:

$$\begin{aligned} (\vec{v}_1)_{CMF} &= g_V^{-1}(\vec{v}_1) = \vec{v}_1 - \vec{V} = \frac{m_2}{M}(\vec{v}_1 - \vec{v}_2) = -\frac{m_2}{M}\vec{v}, \\ (\vec{v}_2)_{CMF} &= g_V^{-1}(\vec{v}_2) = \vec{v}_2 - \vec{V} = \frac{m_1}{M}(\vec{v}_2 - \vec{v}_1) = \frac{m_1}{M}\vec{v}, \end{aligned} \tag{18}$$

where $\vec{v} = \vec{v}_2 - \vec{v}_2$ is the relative velocity. As a result

$$\begin{gathered} (\vec{p}_2)_{CMF} = -(\vec{p}_1)_{CMF} = \\ \frac{m_1 m_2}{M}(\vec{v}_2 - \vec{v}_1) = \frac{m_1 m_2}{M} g_{\vec{v}_1}^{-1}(\vec{v}_2) = -\frac{m_1 m_2}{M} g_{\vec{v}_2}^{-1}(\vec{v}_1). \end{gathered} \tag{19}$$

It is natural to define a new effective particle – *the reduced mass* – with a mass m and a momentum

$$\vec{p} := m\vec{v}, \tag{20}$$

such that

$$\vec{p} = (\vec{p}_2)_{CMF} = -(\vec{p})_{CMF}. \tag{21}$$

Then the comparison of formulas (19) and (20) gives the well known non-relativistic formula for the reduced mass

$$m = \frac{m_1 m_2}{M}. \tag{22}$$

3.2 The Relativistic Two Particle Problem

There exist several different formal approaches to description of the relativistic two-particle system. We will illustrate some of them using the simplest case of two non-interacting particles m_1: $x_1 = \{x_1^0, \vec{x}_1\}$, $p_1 = m_1 u_1$, and m_2: $x_2 = \{x_2^0, \vec{x}_2\}$, $p_2 = m_2 u_2$. Their 4D momenta lie on the hyperboloids

$$2\,{\varphi_1}^{free} := p_1^2 + m_1^2 \stackrel{w}{=} 0, \quad 2\,{\varphi_2}^{free} := p_2^2 + m_2^2 \stackrel{w}{=} 0. \tag{23}$$

1. One can use the two-times formalism and an action of the system:

$$\mathcal{A}_{m_1,m_2} = -\int dt_1\, m_1 \sqrt{1-\vec{v}_1(t_1)^2} - \int dt_2\, m_2 \sqrt{1-\vec{v}_2(t_2)^2}. \tag{24}$$

2. One can develop a 3D+1 approach, which is not transparently covariant. Now we are working in the laboratory frame K using the laboratory time $t = x_1^0 = x_2^0$. Then the system is described by the action

$$\mathcal{A}_{m_1,m_2} = -\int dt \left(m_1 \sqrt{1-\vec{v}_1{}^2} + m_2 \sqrt{1-\vec{v}_2{}^2} \right). \tag{25}$$

3. Finally, one can develop transparently covariant 4D formalism which is in addition invariant under arbitrary local re-parameterizations, using the action

$$\mathcal{A}_{m_1,m_2} = \int d\tau \left(p_1 \dot{x}_1 + p_2 \dot{x}_2 - H \right) \tag{26}$$

with Hamiltonian

$$H := \frac{1}{2\lambda} (\mu_2 {\varphi_1}^{free} + \mu_1 {\varphi_2}^{free}) \tag{27}$$

where λ is a Lagrange multiplier the value of which depends on the choice of the parameter τ, and for the parameters μ_1, μ_2 we have the relation $\mu_1 + \mu_2 = 1$ and some additional conditions [8].

In any approach the problem is obviously invariant under 4D translations, described by the group $Tr(4)_x$ in the flat 4D pseudo-Euclidean space-time $\mathbb{E}_x^{(1,3)}$. As a result of Noether theorem the total momentum

$$P = p_1 + p_2 = const \tag{28}$$

is a conserved quantity. Using this quantity we introduce the relativistic CMF K^U as a frame, which moves relative to the laboratory one with 4D velocity

$$U = P/\sqrt{-P^2} = P/w \tag{29}$$

where w defined by the equation

$$P^2 + w^2 \stackrel{w}{=} 0 \tag{30}$$

plays the role of the CM mass. It is easy to check that this quantity has a right behavior in the non-relativistic limit (NRL):

$$w \xrightarrow{NRL} m_1 + m_2 = M.$$

Following the same procedure, as in the non-relativistic case, and using the rules of Lobachevsky geometry for transition to a new inertial frame – the relativistic CMF, we obtain the formulas:

$$\begin{aligned}
(\vec{u}_1)_{CMF} &= \overrightarrow{\Lambda_U^{-1} u_1} = \\
&\frac{m_2}{w}\left((u_2)_0\vec{u}_1 - (u_1)_0\vec{u}_2 + \frac{(m_1\vec{u}_1 + m_2\vec{u}_2)\times(\vec{u}_2\times\vec{u}_1)}{w + m_1(u_1)_0 + m_2(u_2)_0}\right) = -\frac{m_2}{w}\vec{u}, \\
(\vec{u}_2)_{CMF} &= \overrightarrow{\Lambda_U^{-1} u_2} = \\
&\frac{m_1}{w}\left((u_1)_0\vec{u}_2 - (u_2)_0\vec{u}_1 + \frac{(m_1\vec{u}_1 + m_2\vec{u}_2)\times(\vec{u}_1\times\vec{u}_2)}{w + m_1(u_1)_0 + m_2(u_2)_0}\right) = \frac{m_1}{w}\vec{u},
\end{aligned} \tag{31}$$

which are analogous to the relations (18). Then we see that in the relativistic case the relations (19) are replaced by the analogous ones:

$$(\vec{p}_2)_{CMF} = -(\vec{p}_1)_{CMF} = \frac{m_1 m_2}{w}\overrightarrow{\Lambda_U^{-1} u_2} = -\frac{m_1 m_2}{w}\overrightarrow{\Lambda_U^{-1} u_1} = \frac{m_1 m_2}{w}\vec{u}. \tag{32}$$

where

$$\vec{u} = (u_1)_0\vec{u}_2 - (u_2)_0\vec{u}_1 + \frac{(m_1\vec{u}_1 + m_2\vec{u}_2)\times(\vec{u}_1\times\vec{u}_2)}{w + m_1(u_1)_0 + m_2(u_2)_0}. \tag{33}$$

Now it is natural to define the new effective particle – *the relativistic reduced mass* – with a mass m and a momentum

$$\vec{p} := m\vec{u}, \tag{34}$$

such that

$$\vec{p} = (\vec{p}_2)_{CMF} = -(\vec{p})_{CMF}. \tag{35}$$

Then the comparison of formulas (32) and (34) gives the well known non-relativistic formula for the reduced mass

$$m = \frac{m_1 m_2}{w}. \tag{36}$$

Note that in the non-relativistic case $g^{-1}_{\vec{v}_1}(\vec{v}_2) = g^{-1}_{\vec{v}_2}(\vec{v}_1)$. In contrast, in the relativistic case we have $\overrightarrow{\Lambda^{-1}_{u_1} u_2} \neq -\overrightarrow{\Lambda^{-1}_{u_2} u_1}$, due to the nonlinear character of Lobachevsky velocity space. Nevertheless, we have $(\Lambda^{-1}_{u_1} u_2)_0 = -(\Lambda^{-1}_{u_2} u_1)_0 = (u_1)_0 (u_2)_0 - \vec{u}_1 \cdot \vec{u}_2 = u_0$ and this realation defines u_0 component of the velocity of the relativistic reduced mass. The same result can be obtained independently from the relations (2) and (33):

$$u_0 = \sqrt{1 + \vec{u}^{\,2}} = (u_1)_0 (u_2)_0 - \vec{u}_1 \cdot \vec{u}_2. \tag{37}$$

For components of the momentum of the relativistic reduced mass we obtain from relations (2), (33), (36) and (37):

$$\begin{aligned} p_0 &= \frac{(p_1)_0 (p_2)_0 - \vec{p}_1 \cdot \vec{p}_2}{\sqrt{-(p_1 + p_2)^2}} = -\frac{p_1\, p_2}{\sqrt{-(p_1 + p_2)^2}}, \\ \vec{p} &= \frac{(p_1)_0\, \vec{p}_2 - (p_2)_0\, \vec{p}_1}{\sqrt{-(p_1 + p_2)^2}} + \frac{(\vec{p}_1 + \vec{p}_2) \times (\vec{p}_1 \times \vec{p}_2)}{\sqrt{-(p_1+p_2)^2}\left((p_1)_0 + (p_2)_0 + \sqrt{-(p_1+p_2)^2}\right)}. \end{aligned} \tag{38}$$

In addition we obtain the relations:

$$\begin{aligned} (p_1)^U_0 &= p_0 + \frac{m_1^2}{w}, & (p_2)^U_0 &= p_0 + \frac{m_2^2}{w}; \\ (u_1)^U_0 &= \frac{m_1}{w} + \frac{m_2}{w} u_0, & (u_2)^U_0 &= \frac{m_2}{w} + \frac{m_1}{w} u_0; \\ \vec{u}_{1,2}{}^U &= \mp \frac{m_{1,2}}{w}\, \vec{u} \end{aligned} \tag{39}$$

and

$$u_0 = \frac{w^2 - m_1^2 - m_2^2}{2 m_1 m_2}, \quad p_0 = \frac{1}{2} w - \frac{m_1^2 + m_2^2}{2w}, \tag{40}$$

$$(p_1)_0{}^U = \frac{1}{2}w + \frac{m_1^2 - m_2^2}{2w}, \quad (p_2)_0{}^U = \frac{1}{2}w + \frac{m_2^2 - m_1^2}{2w}. \tag{41}$$

For the parameters μ_1, μ_2 in Eq. (26) one obtains the additional relation [8]:

$$\mu_1 - \mu_2 = \frac{m_1^2 - m_2^2}{w^2} \tag{42}$$

The key formula is:

$$\sqrt{\vec{p}_1{}^2 + m_1^2} + \sqrt{\vec{p}_2{}^2 + m_2^2} = \sqrt{\vec{p}^2 + m^2}. \tag{43}$$

It is valid in the CMF and allows a reduction of the free two-particle relativistic problem to the one effective-particle relativistic problem after exclusion of the CM motion.

It is clear that the formulas for the quantities, related to the effective particle – *"the relativistic reduced mass"*, reflect the properties of the Lobachevsky geometry in velocity space, or in the momentum space.

4 Examples for the Application of the Effective Relativistic Particle

The general procedure for application of the relativistic reduced mass in the relativistic two-particles problems in the case of interacting particles consists in the following steps (See for example [8] and the references therein):

I. In the case of scalar interaction we add the interaction terms $\Phi_{1,2}$ to the constraint functions $\varphi_{1,2}{}^{free}$ in Eq. (23), thus obtaining $\varphi_{1,2} := \varphi_{1,2}{}^{free} + \Phi_{1,2}/2$. Then, using some general procedure, we exclude the CM motion remaining with the effective one-particle relativistic Hamiltonian

$$H := \frac{1}{2\lambda}\left(p^2 + m^2 + \Phi(x)\right). \tag{44}$$

II. In the case of the electromagnetic interactions we use the standard gauge approach, replacing the components of the 4D momentum p_α with $p_\alpha - eA_\alpha(x)$.

III. In the case of the gravitational interaction we replace the pseudo-Euclidean 4D scalar square p^2 with the pseudo-Riemannian one: $p^2 = g^{\alpha\beta}(x)p_\alpha p_\beta$, considering the effective particle as a test particle in an external gravitational field, described by metric $g^{\alpha\beta}(x)$.

In the general case one can combine the scalar, vector and tensor interactions, using simultaneously the above procedures.

For illustration let us consider the following examples:

1. In the article [8] the effective particle Hamiltonian

$$H := \frac{1}{2\lambda}\left({p_r}^2 + \frac{J^2}{r^2} + 1 - \left(\epsilon + \frac{e^2}{r}\right)\right) \tag{45}$$

has been used for the description of relativistic two-particle electromagnetic interaction. Here p_r is the radial momentum, J is the total angular momentum, $e^2 = -e_1e_2$ is related with the electric charges $e_{1,2}$, and ϵ is the dimensionless effective-particle energy. It was shown that this Hamiltonian reproduces the right relativistic effects: the orbit, the perihelion shift, etc. produced by standard approximate relativistic Hamiltonian up to the terms of order $\frac{1}{c^2}$.

2. As shown article [8] the effective particle Hamiltonian in Schwarzschild gravitational field:

$$H := \frac{1}{2\lambda}\left(\left(1 - \rho\frac{J}{r}\right){p_r}^2 + \frac{J^2}{r^2} + 1 - \frac{\epsilon^2}{1-\rho\frac{J}{r}}\right) \tag{46}$$

describes in a correct way the general relativistic two-particles gravitational interaction up to the terms of order $\frac{1}{c^2}$. In particular the orbit, the perihelion shift and a new formula

$$\omega^* = \frac{\omega_S^*}{1-\nu|\varepsilon^*|} > \omega_S^* \tag{47}$$

for the angular frequency of the last stable orbit (LSO) was derived. Here $\rho = 2m_1m_2G/J$, $\nu = \frac{m_1m_2}{(m_1+m_2)^2}$, $\varepsilon^* = -\frac{1}{\nu}\left(1-\sqrt{1-2\nu\left(1-\sqrt{8/9}\right)}\right)$, and ω_S^* is the Schwarzschild value of the frequency of LSO.

3. In the article by Crater and Yang (see in [7]) the effective one-particle approach for the Wheeler-Feynman action-at-a-distance of the relativistic two-particle system

with scalar and electromagnetic interaction was examined. The final result is that if one wishes to reproduce the right relativistic results up to the order $\frac{1}{c^4}$, using the effective one-particle Hamiltonian, then one has to introduce in this Hamiltonian the following modified electromagnetic and scalar potentials:

$$\begin{aligned} eA_0 &:= \frac{e_1e_2}{r} + \frac{5(e_1e_2)^2(g_1g_2 - e_1e_2)}{12m_1m_2r^3}, \\ e\vec{A} &:= \frac{5(e_1e_2)^2(g_1g_2 - e_1e_2)}{12m_1m_2r^3}\vec{p}, \\ \Phi &:= -\frac{g_1g_2}{r} - \frac{5(g_1g_2)^2(g_1g_2 - e_1e_2)}{12m_1m_2r^3}. \end{aligned} \tag{48}$$

Here $g_{1,2}$ are the "scalar charges" of the particles.

4. An analogous result was derived by Buonanno and Damour (see in [7]) for general relativistic two-body problem. They have introduced an "effective" metric of the form

$$\begin{aligned} ds^2{}_{eff} = &\left(1 + \frac{a_1}{r_{eff}} + \frac{a_2}{r_{eff}^2} + \frac{a_3}{r_{eff}^3} + \dots\right) dt_{eff}^2 - \\ &\left(1 + \frac{b_1}{r_{eff}} + \frac{b_2}{r_{eff}^2} + \frac{b_3}{r_{eff}^3} + \dots\right) dr_{eff}^2 - \\ &\left(1 + \frac{c_1}{r_{eff}} + \frac{c_2}{r_{eff}^2} + \frac{c_3}{r_{eff}^3} + \dots\right) r_{eff}^2 \left(d\theta_{eff}^2 + \sin^2\theta_{eff} d\varphi_{eff}^2\right) \end{aligned} \tag{49}$$

with a proper coefficients $a_{1,2,3\dots}$, $b_{1,2,3\dots}$, $c_{1,2,3\dots}$ to reach the standard 2PN approximation results.

5. In the very recent articles by Faruque and coauthors (see in [9]) the relativistic effective particle approach was applied to the gravitational interaction of two relativistic spinning particles using Kerr metric, instead of Schwarzschild one. The corrections to the orbits and to the perihelion shift:

$$\delta\varphi = 2\pi\left(\frac{1}{\sqrt{\gamma}} - 1\right) + \frac{3\pi}{2}\frac{\alpha\rho}{\sqrt{\gamma}\gamma^2} \tag{50}$$

were obtained. Here $\alpha = 1 - 2\mu\epsilon + 3\mu^2\epsilon^2, \gamma = 1 - 3\beta\mu^2,\ \beta = 1 - \epsilon$, and $\mu = a/J$, where a is the specific spin of the Kerr metric.

5 Conclusion

The above examples give a strong indications that the application of the method of the relativistic effective one-particle to the relativistic two particle problem is a powerful tool for the study of modern topics in physics. This method is based on the Lobachevsky geometry of the relativistic velocity and momentum spaces. It deserves further development as a fundamental theoretical approach to the relativistic problems.

Acknowledgments

The author is deeply grateful to the Organizing Committee of the International seminar: *Application and Development of Lobachevsky Ideas in Modern Physics*, dedicated to 75-th birthday of Professor N. A. Chernikov, Dubna, 25-27 February 2004 for the invitation to attend this meeting and for the possibility to give this talk.

References

[1] V. A. Fock, *The Theory of Space, Time and Gravitation*, Pergamon, Oxford, 1964.

[2] L. D. Landau, E. M. Lifshitz, *The Classical Theory of Fields*, (2d ed.; Reading, Mass: Addison-Wesley, 1962).

[3] J. A. Wheeler, R. P. Feynman, Rev. Mod. Phys. **21**, 425 (1949).

[4] E. H. Kerner, *The Theory of Action-at-a-Distance in Realtivistic Particle Dynamics*, Gordon and Breach Sci. Publ., NY, 1972.

[5] B. M. Barbashov, N. A. Chernikov, *Classical Dynamics of the Relativistic Strings*, preprint JINR R2-7852, Dubna, 1974 (in Russian); B. M. Barbashov, V. V. Nesterenko, *A Model of Relativistic String and the Adron Physics*, Ener-

goatomizdat, Moscow, 1987 (in Russian); B. M. Barbashov, V. V. Nesterenko, *Introduction to the Relativistic String Theory*, Singapore, World Scientific, 1990.

[6] N. A. Chernikov, N. S. Shavohina, preprint JINR, Dubna, R2-10375, 1977; Theor. Math. Phys. **42**(1), p. 59 (1980); **43**(3), p. 356 (1980). N. S. Shavohina, Izv. Vuzov, Fizika, **7**, p. 91 (1981); DAN SSSR, **265**(4), p. 852 (1983); Izv. Vuzov, Fizika, **7**, p. 66 (1982); Izv. Vuzov, Fizika, **7**, p. 46 (1983); Problemy teorii gravitacii i elementarnyh chastic, **15**, Energoatomizdat, Moscow, p. 141, 1985; Problemy teorii gravitacii i elementarnyh chastic, **16**, Energoatomizdat, Moscow, p. 189, 1985; Problemy teorii gravitacii i elementarnyh chastic, **17**, Energoatomizdat, Moscow, p. 187, 1986; Mezhdunarodnoe Soveschanie po problemam kvantovoj teorii polia, Dubna, p. 246, 1987. (All references are in Russian.)

[7] I. T. Todorov, Phys. Rev. D**3**, 2351 (1971).

I. T. Todorov, *Quasipotential approach to the two-body problem in quantum field theory*, in: Properties of Fundamental Interactions at High Energy, ed. A. Zichichi (Ed. Compositori, Bologna 1973), 951.

V. A. Rizov, I. T. Todorov, B. L. Aneva, Nucl. Phys. B**98**, 447 (1975).

M. Malvetti, H. Pilkuhn, Phys. Rep. **248**, 1 (1994).

I. T. Todorov, *Dynamics of relativistic point particles as a problem with constraints*, Commun. JINR, E-2-10125, Dubna (1976).

A. Maheshwari, E. R. Nissimov, I. T. Todorov, Lett. Math. Phys. **5**, 359 (1981).

H. Crater, D. Yang, *A covariant extrapolation of the noninvariant two particle Wheeler-Feynman Hamiltonian from Todorov equation and Dirac's constraint mechanics*, J. Math. Phys. **32**, 2374 (1991).

A. Buonanno, T. Damour, *Effective one-body approach to general relativistic two-body dynamics*, gr-qc/981100, Phys.Rev. D**59**, 084006 (1999); **62**, 064015 (2000).

[8] Fiziev P.P., Todorov I.T., *Effective one-body approach to the relativistic two-body problem*, E-print: gr-qc/0010104; Phys. Rev. D**63**, 104007 (2001).

[9] S. Faruque, *Relativistic Periastron Shift of a Particle in Kerr Field: A Particular Case-Study*, Acta Phys. Slovaca, (2003).

F.De. Polis, G. Ingrosso, A. Nucita, A. Zakhariev, astro-ph/0310213.

T. Damour, Phys. Rev. D**64**, 124013 (2001), gr-qc/0103018.

Faruque S. B., Ahsan M. A., Ishwar B., Phys. Lett. A312, 166 (2003).

Faruque S. B., *Circular Orbits of a Particle in Kerr Field: A study using effective one-body approach* , Acta Phys. Slovaca, **53**, 157 (2003).

[10] P. M. A. Dirac, Canad. J. Math. **2**, 129 (1950); Proc. Roy. Soc. A**246**, 326 (1958).

I. T. Todorov, *Constraint hamiltonian mechanics of directly interacting relativistic particles*, in: Relativistic Action at a Distance. Classical and Quantum Aspects, ed. J. Llosa, Lecture Notes in Physics **162** (Springer, Berlin, 1982), 213.